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ON THE REDUCTION OF INTEGRAL EQUATIONS OF THE THEORY OF ELASTICITY TO INFINITE SYSTEMS

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We indicate formal methods for the reduction of the integral equations of the theory of elasticity (not considered in [1]) to infinite systems of algebraic equations. We consider an integral equation of the first kind with a difference kernel of mixed type, i. e. containing both Fredholm and Volterra operators. To such an equation one can reduce, for example, the problem of the bending of a semi-infinite plate on a linearly deformable foundation when for the inversion of the differential operator one makes use of the Cauchy function rather than the Green function [2]. The method by which this equation is reduced to an infinite system is based on the presence of spectral relations for the semi-infinite interval. In addition to the relations of similar type, indicated in [3], new spectral relations on the semi-infinite interval are constructed. An integral equation of the second kind and of mixed type is considered. Integral equations of the first and second kind with difference kernels and data prescribed on the axis with a cut-off segment are studied. We consider an integral equation of the second kind on a finite interval with a kernel represented through an improper integral of the product of Bessel functions. The suggested methods can be carried over to the corresponding systems of integral equations.

1. Let us consider the integral equation of mixed type

$$\int_0^\infty k(x-y)\varphi(y)\,dy + \int_0^x l(x-y)\varphi(y)\,dy = f(x) \tag{1.1}$$

We will assume that the integral representation

$$k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(t)\,e^{-ixt}\,dt \tag{1.2}$$

and the asymptotics

$$K(t) = \gamma t^{2\mu-1}[1 + O(t^{-1})], \qquad t \to \infty \quad (|\mu| < {}^1/_2) \tag{1.3}$$

are valid. This allows us to represent the function $k(x)$ in the form

$$k(x) = \gamma k_\mu(x) + d(x) \tag{1.4}$$

where the first term, carrying a singularity, has the form

$$k_\mu(x) = \frac{2^\mu K_\mu(|x|)}{\Gamma(^1/_2 - \mu)\sqrt{\pi}\,|x|^\mu} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ixt}\,dt}{(1+t^2)^{1/2-\mu}} \tag{1.5}$$

($K_\mu(z)$ is the Macdonald function), while the second term can be represented in the form

$$d(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} D(t)\,e^{-ixt}dt, \qquad D(t) = K(t) - \frac{\gamma}{(1+t^2)^{1/2-\mu}} \tag{1.6}$$

In order to reduce the integral equation (1.1) to an infinite system of equations, we make use of a scheme applied in [1] for the simpler case when $l \equiv 0$. According to this scheme, we construct the solution in the form of the following expansion in Chebyshev-Laguerre polynomials:

$$\varphi(x) = \frac{2e^{-x}}{x^{1/2-\mu}}\sum_{m=0}^{\infty}\frac{\varphi_m}{\mu_m}L_m^{\mu-1/2}(2x) \qquad \left(\mu_m = \frac{\Gamma(^1/_2 + \mu + m)}{2^{1/2-\mu}\,m!}\right) \tag{1.7}$$

By carrying out subsequently the operations indicated in [1] and based essentially on the use of the spectral relation [4]

$$\int_0^\infty k_\mu(x-y)\,e^{-y}y^{\mu-1/2}L_m^{\mu-1/2}(2y)\,dy = \mu_m e^{-x}L_m^{\mu-1/2}(2x) \tag{1.8}$$

we obtain the following infinite system of algebraic equations:

$$\varphi_n + \frac{1}{\gamma}\sum_{m=0}^{\infty} a_{n-m}\varphi_m + \frac{1}{\gamma}\sum_{m=0}^{\infty} b_{n-m}\varphi_m = \frac{f_n}{\gamma} \qquad (n = 0, 1, 2, \ldots) \tag{1.9}$$

$$a_k = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{D^*(t)}{1+t^2}\left(\frac{t+i}{t-i}\right)^k dt = \frac{1}{2\pi}\int_0^{2\pi} D^*\left(-\operatorname{ctg}\frac{\varphi}{2}\right)e^{-ik\varphi}\,d\varphi$$

$$(D^*(t) = (1+t^2)^{1/2-\mu}K(t) - \gamma)$$

$$b_k = \frac{1}{\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{L(p)}{(1-p^2)^{1/2+\mu}}\left(\frac{p+1}{p-1}\right)^k dp, \qquad f_n = \frac{1}{\mu_n}\int_0^\infty \frac{f(x)\,L_n^{\mu-1/2}(2x)\,dx}{e^x x^{1/2-\mu}} \tag{1.10}$$

By $L(p)$ we denote the Laplace transform of the function $l(x)$, i.e.

$$L(p) = \int_0^\infty e^{-px}l(x)\,dx$$

In this connection one has to take into account, that in contrast to [1], after the substitution of (1.7) into (1.1) and term by term integration one has to make use of the convolution theorem for the Laplace transforms. In order to avoid operating on divergent integrals in subsequent integrations, we have to impose a restriction on the growth of the function $l(x)$, i.e.

$$l(x) = o(e^x), \qquad x \to \infty \tag{1.11}$$

This allows us to consider the parameter ε, contained in the formula for b_k, smaller than unity but larger than the real parts of the numbers which determine the singular points of the Laplace transform of the function $l(x)$. As it follows from the formulas (1.10) for a_k, they represent the Fourier coefficients of the function $D^*(-\mathrm{ctg}^1/_2\varphi)$ and therefore for their computation we can apply the known formulas of trigonometric interpolation [5]. These same formulas can be applied also for the computation of the remaining coefficients of the infinite system (1.9) if we require in addition that the function $l(x)$ be absolutely integrable on the semiaxis ($\varepsilon = 0$) and that we should have the integral representation

$$f(x) = \frac{1}{2\pi} \int\limits^{\infty} F(t) e^{-itx} dt \tag{1.12}$$

Then, assuming $\varepsilon = 0$ and $p = i \,\mathrm{ctg}\, {}^1/_2\varphi$ in the second formula of (1.10) and substituting (1.12) into the third formula of (1.10), we obtain

$$b_k = \frac{1}{2\pi} \int\limits_0^{2\pi} L_\mu\left(i \,\mathrm{ctg}\, \frac{\varphi}{2}\right) e^{-ik\varphi} d\varphi, \qquad L_\mu(z) = \frac{L(z)}{(1-z^2)^{\mu - 1/2}} \tag{1.13}$$

$$f_n = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{F(u)}{(1+iu)^{\mu+1/2}} \left(\frac{u+i}{u-i}\right)^n du = \frac{1}{2\pi} \int\limits_0^{2\pi} F_\mu\left(-\mathrm{ctg}\, \frac{\varphi}{2}\right) e^{-in\varphi} d\varphi$$

$$F_\mu(z) = 2(1-iz)(1+iz)^{1/2-\mu} F(z)$$

The infinite systems of type (1.9) admit, as it is known [6], an exact solution obtained by the method of factorization on the unit circumference. To this end, one has to sum the series

$$\sum_{k=-\infty}^{\infty} (a_k + b_k) z^k = A(z) \tag{1.14}$$

and to factor out the function $[1 + \gamma^{-1} A(z)]^{-1}$ on the unit circumference. In the case under consideration, by virtue of formulas (1.10) and (1.13), the series in the left-hand side of (1.14) can be easily summed for $z = e^{i\varphi}$ and we obtain

$$A(z) = D^*\left(i \frac{1+z}{1-z}\right) + L_\mu\left(\frac{1+z}{1-z}\right) \tag{1.15}$$

In [6] formulas are given by means of which the factorization problem is solved on the unit circumference. The existence and uniqueness of an exact solution in one or another class of functions [6] is guaranteed by the conditions

$$\sum_{k=-\infty}^{\infty} |a_k + b_k| < \infty, \qquad \gamma + D^*(-\mathrm{ctg}\, {}^1/_2\varphi) + L_\mu(i \,\mathrm{ctg}\, {}^1/_2\varphi) \neq 0 \tag{1.16}$$

$$(0 \leqslant \varphi \leqslant 2\pi)$$

$$\{\arg [D^*(-\mathrm{ctg}\, {}^1/_2\varphi) + L_\mu(i \,\mathrm{ctg}\, {}^1/_2\varphi)]\}_{\varphi=0}^{\varphi=2\pi} = 0$$

The exact solution of system (1.9) obtained by the method of factorization reduces the number of quadratures in the exact solution of the initial equation in comparison with the one obtained without passing to the system (1.9), however, frequently it turns out that numerical evaluation is difficult. Therefore in many cases it is more convenient to obtain an approximate solution of the infinite system by the method of reduction [truncation]. Moreover, as shown in [7], conditions (1.16) are necessary and sufficient for its applicability.

The present method of reduction of the integral equation (1.1) to the infinite system (1.9) is connected in an essential manner with the spectral relation (1.8) on the semi-infinite interval. Its use became possible due to the asymptotics (1.3). If the asymptotics for $K(t)$ is different, then one has to make use of another spectral relation. In order to have this possibility, it is necessary to have available a sufficiently large collection of spectral relations on the semi-infinite interval.

2. We construct a new series of spectral relations on a semi-infinite interval. We will proceed from the following relation [3]:

$$\int_0^1 \frac{\Pi^*(x, y)}{(1-y)^{1-\alpha}} P_n^{\alpha-1,\, 1+\rho-\alpha}(2y-1)\, dy = \frac{\sigma_n P_n^{\alpha-1,\, 1+\rho-\alpha}(2x-1)}{x^{\alpha-\rho-1}} \tag{2.1}$$

$$\Pi^*(x, y) = \int_0^z \frac{s^\rho\, ds}{(x-s)^\alpha (y-s)^\alpha}, \qquad \sigma_n = \frac{\Gamma^2(1-\alpha)\,\Gamma(\alpha+n)\,\Gamma(1+\rho+n)}{n!\,\Gamma(2+\rho-\alpha+n)}$$

$$(z = \min(x, y);\ \operatorname{Re}(1+\rho,\ 1-\alpha) > 0)$$

The polynomial kernel $\Pi^*(x, y)$ for $x < y$, if we take into account the known integral representation [8] for the Gauss function $F(a, b; c; x)$, can be represented in the form

$$\Pi^*(x, y) = \boldsymbol{B}(\rho+1, 1-\alpha)\, y^{-\alpha} x^{1+\rho-\alpha} F(\alpha, 1+\rho; 2-\alpha+\rho; x/y)$$

In the case $y < x$ one has to interchange the places of x and y. If in (2.1) we perform now the change of variables $x = e^{-\xi}$ and $y = e^{-\eta}$, then we arrive at the following spectral relation on the semi-infinite interval:

$$\int_0^\infty \frac{e^{-\omega|\xi-\eta|} F(\alpha, 2\omega; 1-\alpha+2\omega; e^{-|\xi-\eta|}) P_n^{\alpha-1,\, 2\omega-\alpha}(2e^{-\eta}-1)\, d\eta}{e^{(1+\omega-\alpha)\eta}(1-e^{-\eta})^{1-\alpha}} = \tag{2.2}$$

$$= \frac{\pi(\alpha)_n (2\omega)_n e^{-\omega\xi}}{\sin\pi\alpha n!\,(1-\alpha+2\omega)_n} P_n^{\alpha-1,\, 2\omega-\alpha}(2e^{-\xi}-1)$$

$$(2\omega = 1+\rho,\ n = 0, 1, 2, \ldots)$$

From the obtained spectral relation, making use of the transformation formulas for the Gauss function and changing the parameters, one can obtain a series of spectral relations, a great many of which can be applied to equations of the type (1.1). We indicate those for which the Gauss function degenerates into an elementary function. Assuming, for example, $\omega = \alpha - {}^1/_2$, instead of (2.2), we obtain

$$\int_0^\infty \left|\operatorname{sh}\frac{\xi-\eta}{2}\right|^{1-2\alpha} \frac{C_n^{\alpha-1/2}(2e^{-\eta}-1)\, d\eta}{e^{1/2\eta}(1-e^{-\eta})^{1-\alpha}} = \frac{2^{2\alpha-1}(2\alpha-1)_n}{\sin\pi\alpha n!\, e^{(\alpha-1/2)\xi}} C_n^{\alpha-1/2}(2e^{-\xi}-1) \tag{2.3}$$

Here we have made use of the known relation between the Jacobi polynomials and the Gegenbauer polynomials $C_n^\nu(z)$ [8].

The spectral relation

$$\frac{1}{\pi}\int_0^\infty \ln\left|\operatorname{cth}\frac{\xi-\eta}{4}\right| \frac{P_n^{1/2,\, -1/2}(1-2e^{-\eta})}{e^{1/2\eta}(1-e^{-\eta})^{1/2}}\, d\eta = \frac{P_n^{1/2,\, -1/2}(1-2e^{-\xi})}{(1/2+n)\, e^{1/2\xi}} \tag{2.4}$$

follows from (2.2) for $\alpha = \omega = {}^1/_2$. The Jacobi polynomials contained in (2.4) can be replaced by Chebyshev polynomials of the first kind $T_n(z)$, or of the second kind $U_n(z)$, by making use of the relations

$$n!\, P_n^{1/2,\, -1/2}(1-2z) = (1/2)_n\, U_{2n}\left(\sqrt{1-z}\right) = (-1)^n (1/2)_n\, z^{-1/2} T_{2n+1}\left(\sqrt{z}\right) \quad (2.5)$$

The spectral relation (2.4) can be applied to the solving of Eq. (1.1), if in the asymtotics (1.3) we have $\mu = 0$. In this case it is easy to isolate the singular part from the kernel function (1.2)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\operatorname{th} \pi t}{t} e^{-ixt}\, dt = \frac{1}{\pi} \ln \left| \operatorname{cth} \frac{x}{4} \right| \tag{2.6}$$

In [9], the spectral relation (2.5) is obtained in a different manner and in the same place an application of it is given to the solving of an integral equation of type (1.1) for $l(x) \equiv 0$. Spectral relations of another type on the semi-infinite interval are obtained if we start from the relations [10]

$$\int_0^1 \frac{W_{\mu,\gamma}^{\nu}(x, y)\, P_m^{\gamma,\, -\sigma_+}(1-2y^2)\, dy}{(1-y^2)^{\sigma_+}\, y^{-\gamma-1}} = \frac{\lambda_m}{x^{-\mu}} P_m^{\mu,\, -\sigma_-}(1-2x^2) \tag{2.7}$$

$$\lambda_m = \Gamma(1-\sigma_+ + m)\, \Gamma(\sigma + m)\, 2^{\nu-1} [m!\, \Gamma(1+\mu+m)]^{-1}$$
$$(0 \leqslant x \leqslant 1,\ 2\sigma_{\pm} = 1 - \nu \pm (\gamma - \mu),\ 2\sigma = 1 + \nu + \gamma + \mu,\ \operatorname{Re} \sigma_{\pm} < 1)$$

Here $W_{\mu,\gamma}^{\nu}(x, y)$ is the discontinuous Weber-Sonine integral, expressed in terms of Bessel functions by the formula

$$W_{\mu,\gamma}^{\nu}(x, y) = \int_0^{\infty} t^{\nu} J_{\mu}(tx)\, J_{\gamma}(ty)\, dt \tag{2.8}$$
$$(\operatorname{Re}(1+\nu+\gamma+\mu) > 0,\ \operatorname{Re} \nu < 1)$$

Making use of its known representation in terms of the Gauss' hypergeometric function [8], we find the following important property:

$$W_{\mu,\gamma}^{\nu}(\xi^{-1}, \eta^{-1}) = (\xi\eta)^{1+\nu} W_{\gamma,\mu}^{\nu}(\xi, \eta) \tag{2.9}$$

Performing in (2.7) the substitution $x = \xi^{-1}$ and $y = \eta^{-1}$ and making use of property (2.9), we obtain a spectral relation of the semi-infinite interval

$$\int_1^{\infty} \frac{W_{\mu,\gamma}^{\nu}(\xi, \eta)\, P_m^{\gamma,-\sigma_+}(1-2\eta^{-2})\, d\eta}{\eta^{1+\rho}(\eta^2-1)^{\sigma_+}} = \frac{\lambda_m P_m^{\mu,-\sigma_-}(1-2\xi^{-2})}{\xi^{1+\nu+\mu}} \tag{2.10}$$
$$(2\rho = 1 - \nu + \gamma + \mu,\ 1 \leqslant \xi < \infty)$$

Giving different values to the parameters ν, γ, μ, we can obtain from this relation a series of spectral relations. We indicate only one of them

$$\int_1^{\infty} K\left(\frac{2\sqrt{\xi\eta}}{\xi+\eta}\right) \frac{P_{2m}\left(\sqrt{1-\eta^{-2}}\right) d\eta}{(\xi+\eta)\, \eta^{3/2} \sqrt{\eta^2 - 1}} = \left[\frac{\pi}{2} \frac{(2m-1)!!}{2m!!}\right]^2 P_{2m}\left(\sqrt{1-\xi^{-2}}\right) \tag{2.11}$$

obtained from (2.10) for $\nu = \gamma = \mu = 0$. Here $K(x)$ is the complete elliptic integral of the first kind and $P_m(z)$ is Legendre's polynomial. The spectral relations of the type (2.10) allow to reduce, in the described manner (see Sect. 1), to infinite systems the integral equations of the first kind given on a semi-infinite interval, whose kernels do not depend on the difference of the arguments but have the same singularity as the kernels in (2.10) and (2.11). Integral equations of this type occur in the problems of the concentration of stresses around circular cracks.

3. The method of reduction of the integral equation (1.1) to an infinite system can be easily carried over to the corresponding equation of the second kind

$$\varphi(x)+\int_0^\infty k(x-y)\varphi(y)\,dy+\int_0^x l(x-y)\varphi(y)\,dy=f(x) \tag{3.1}$$

In this connection it is not necessary to have a spectral relation on the semi-infinite interval and we can use arbitrary functions, orthogonal on the semi-infinite or on a finite (*) interval, provided the weight functions are not equal to infinity at the endpoints of the orthogonality interval. However, obviously, the application of the Laguerre polynomials, i. e. the representation of the solutions in the form

$$\varphi(x)=2e^{-x}\sum_{m=0}^\infty \varphi_m L_m(2x) \tag{3.2}$$

allows us to obtain the infinite system (**)

$$\varphi_n+\sum_{m=0}^\infty a_{n-m}\varphi_m+\sum_{m=0}^n b_{n-m}\varphi_m=f_n \tag{3.3}$$

which is the discrete analog of the initial equation. In addition, for the coefficients b_k, f_n we have the previous formulas (1.10) in which one has to put $\mu={}^1/_2$. Thus, the ramification points in the integral, which determine b_k, disappear and this, by virtue of the regularity of $L(z)$ in the semiplane $\operatorname{Re} z>\varepsilon>1$, leads to the fact that $b_k=0$ and $k<0$. For the coefficients a_k we also have formula (1.10) where $D^*(t)$ has to be replaced by $K(t)$. If we consider, as above, that $l(x)$ is absolutely integrable on the semiaxis and that $f(x)$ is representable in the form (1.12), then formula (1.13) holds for $\mu={}^1/_2$ also for the case under consideration.

The conditions (1.16) for $\gamma=1$, $D^*=K$ and $L_\mu=L$ are necessary and sufficient for the applicability of the reduction method to the infinite system (3.3).

It is useful to remark that in the case of the Volterra equation $(k(x)\equiv 0)$ the infinite system (3.3) $(a_k=0)$ degenerates into recursion relations for the unknown φ_m For them one can give also an explicit expression in terms of f_n [11].

Thus, we have outlined another method for obtaining the exact solution of the Volterra integral equation without making use of the inversion formula for the Laplace transform. If for the Volterra equation

$$\varphi(x)+\int_x^\infty l(x-y)\varphi(y)\,dy=f(x) \qquad (x\geqslant 0) \tag{3.4}$$

we look for a solution in the form (3.2), then it can be also reduced to the infinite system

$$\varphi_n+\sum_{m=n}^\infty b_{n-m}\varphi_m=f_n \tag{3.5}$$

which is the discrete analog of the initial equation. The formulas for the coefficients of this system are similar to those corresponding to the coefficients of the system (3.3).

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*) In this connection, by a suitable change of variables one has to reduce the finite interval to a semi-infinite one.

**) This, relative to the case $l(x)\equiv 0$, was, apparently, discovered for the first time in [7].

In the case of the integral equation of the form

$$\varphi(x)+\int_0^\infty k(x-y)\,w(y)\,\varphi(y)\,dy+\int_0^x l(x-y)\,w(y)\,\varphi(y)\,dy=f(x) \quad (3.6)$$

one can proceed in the following manner. As before, we look for the solution in the form (3.3), but in addition we assume

$$\psi(x)=w(x)\,\varphi(x)=2\sum_{m=0}^{\infty} e^{-x}\psi_m L_m(2x)$$

As a result we arrive at the following infinite system:

$$\varphi_n+\sum_{m=0}^{\infty} a_{n-m}\psi_m+\sum_{m=0}^{n} b_{n-m}\psi_m=f_n \quad (3.7)$$

$$\psi_n=\sum_{m=0}^{\infty} b_{n,m}\varphi_m,\quad b_{n,m}=2\int_0^\infty e^{-2x}w(x)\,L_n(2x)\,L_m(2x)\,dx=$$

$$=\sum_{k=0}^{n+m} c_k w_k,\quad w_k=\int_0^\infty e^{-k}x^k w\left(\frac{x}{2}\right)dx,\ c_k=\sum_{r=0}^{k}\frac{(-n)_r\,(-m)_{k-r}}{r!^2\,(k-r)!^2}$$

In addition, the coefficients a_k, b_k, f_k are determined by the same formulas as the corresponding coefficients of the infinite system (3.3).

4. We indicate a method of reduction to an infinite system the integral equation of the form

$$\left(\int_{-\infty}^{-a}+\int_a^\infty\right)k(x-y)\,\varphi(y)\,dy=f(x) \qquad (|x|>a) \quad (4.1)$$

We will assume that from the kernel one can isolate a singular part $k_*(x-y)$, for which we have the spectral relation

$$\int_0^\infty k_*(x-y)\,p(y)\,\pi_n{}^*(y)\,dy=\sigma_n g(x)\,\pi_n{}^*(x) \qquad (x\geqslant 0,\ n=0,1,2\ldots) \quad (4.2)$$

Here (*) $\pi_n{}^*(x)=\pi_n[\rho(x)]$, where $\pi_n(x)$ is a polynomial, orthogonal in the sense that

$$\int_0^\infty p(x)\,g(x)\,\pi_n{}^*(x)\,\pi_m{}^*(x)\,dx=\lambda_n\delta_{mn} \quad (4.3)$$

Thus, let

$$k(z)=\gamma k_*(z)+d(z) \qquad (\gamma=\text{const}) \quad (4.4)$$

We introduce the function

$$f_\pm(x)=f(\pm x),\quad \varphi_\pm(x)=\varphi(\pm x) \qquad (x>a) \quad (4.5)$$

and we write Eq. (4.1) as two equations: one for $x>a$, and the other for $x<-a$. Then, as a result of obvious changes of variables and of the use of (4.4) and (4.5), we

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*) In order to simplify the writing, we assume that the singular part of the kernel function has the property $k_*(x)=k_*(-x)$. In [3] spectral relations on the semi-infinite interval indicated also for the asymmetric kernel, therefore we apply the tentative method also in the case when $k_*(x)\neq k_*(-x)$, but then it will be necessary to introduce two families of π^*-polynomials.

arrive at the systems

$$\gamma\int_a^\infty k_*(x-y)\varphi_\pm(y)\,dy+\int_a^\infty d(\pm x\mp y)\varphi_\pm(y)\,dy+\int_a^\infty k(\pm x\pm y)\varphi_\mp(y)\,dy=$$
$$=f_\pm(x)\quad(x\geqslant a)\tag{4.6}$$

According to the spectral relation (4.2) we construct the solution of the system (4.6) in the form

$$\varphi_\pm(x)=\sum_{m=0}^\infty\frac{\varphi_m{}^\pm}{\sigma_m}p(x-a)\pi_m{}^*(x-a)\tag{4.7}$$

We substitute (4.7) into (4.6) and then we multiply both sides of Eq. (4.6) by $\lambda_n{}^{-1}p(x-a)\ \pi_n{}^*\ (x-a)$ and we integrate over the interval $(a,\ \infty)$. Making use of (4.2) and (4.3) we arrive at the infinite systems

$$\gamma\varphi_n{}^\pm+\sum_{m=0}^\infty a_{nm}^\pm\varphi_m{}^\pm+\sum_{m=0}^\infty b_{nm}^\mp\varphi_m{}^\mp=f_n{}^\pm\qquad(n=0,1,2,\ldots)\tag{4.8}$$

$$\frac{\{a_{nm}^\pm,\ b_{nm}^\mp\}}{\lambda_n^{-1}\sigma_m^{-1}}=\int_0^\infty\int_0^\infty\{d(\pm\xi\mp\eta),\ k(\pm 2a\pm\xi\pm\eta)\}\frac{\pi_n{}^*(\xi)\pi_m{}^*(\eta)\,d\xi\,d\eta}{[p(\xi)\,p(\eta)]^{-1}}$$

$$f_n{}^\pm=\frac{1}{\lambda_n}\int_0^\infty f^\pm(a+\xi)\,p(\xi)\,\pi_n{}^*(\xi)\,d\xi$$

Obviously, in the case $d(x)=d(-x)$ the infinite systems degenerate into two independently solvable systems (one for the even component of $f(x)$, and the other for the odd component). The problem of the bending of two semi-infinite plates resting on a linearly deformable foundation [2] and also some plane problems of the theory of cracks can be reduced to equation (4.1) with a kernel which can be represented in the form (1.4). This allows us to take (1.8) as the spectral relation and to set in the formulas (4.7) and (4.8)

$$p(x)=e^{-x}x^{\mu-1/2},\quad g(x)=e^{-x},\quad \pi_n{}^*(x)=2^\mu L_n^{\mu-1/2}(2x)$$
$$\sigma_n=\lambda_n=\mu_n,\qquad k^*(x)=k_\mu(x)\tag{4.9}$$

The coefficients of the infinite system become essentially simpler having the form

$$a_{nm}^+=d_{n-m},\quad a_{nm}^-=d_{m-n},\quad b_{nm}^\pm=c_{n+m}^\pm\tag{4.10}$$

$$d_k=\frac{(-1)^k}{\pi}\int_{-\infty}^\infty\frac{D(t)}{(1+t^2)^{1/2+\mu}}\left(\frac{1-it}{1+it}\right)^k dt$$

$$c_k^\pm=\frac{(-1)^k}{\pi}\int_{-\infty}^\infty\frac{K(\pm t)\,e^{-2ait}}{(1+it)^{1+2\mu}}\left(\frac{1-it}{1+it}\right)^k dt$$

For some particular functions $D(t)$ and $K(t)$ these integrals can be reduced by the methods of contour integration to known special functions [4]. In the general case, the use of the method of trigonometric interpolation [5, 12] is the most convenient. This is equivalent to the approximation, for example, of the function $D(t)$ in the form [12]

$$D(t)\approx 2\sum_{l=-N}^N d_l{}^*\left(\frac{1+it}{1-it}\right)^l\frac{(-1)^l}{1-it}\tag{4.11}$$

The formulas for the approximation coefficients $d_l{}^*$ are given in [12, 5]. They are expressed in terms of the discrete values of the function $D(t)$. Making use of (4.11) we can obtain the formula

$$d_k \simeq \frac{2^{1-2\mu}\,\Gamma(2+2\mu)}{(-1)^k(1+2\mu)} \sum_{l=-N}^{N} \frac{(-1)^l d_l{}^*}{\Gamma(1/2+k-l+\mu)\,\Gamma(3/2+l-k+\mu)}$$

A formula of similar structure can be obtained by the same method for the coefficients $c_k{}^{\pm}$. In this connection it is useful to take into account that by virtue of (1.6)

$$K(t) = \gamma(1+t^2)^{\mu-1/2} + D(t)$$

For the most frequently encountered case $\mu = 0$ the solution can be expressed in terms of Hermite polynomials of even index [4]. In addition, in this case we can apply the spectral relation (2.4) and the solution can be obtained in the form of a series in Chebyshev polynomials. The described method can be easily carried over also to the corresponding equations of the second kind. In this connection, the necessity in the spectral relation is eliminated (see Sect. 3) and the use of the Laguerre polynomials $L_n(2x)$ turns out to be the most natural.

5. In [13] the dual equation

$$\int_0^\infty \xi^{-2\alpha}\,[1+K(\xi)]\,\chi(\xi)\,J_\nu(\xi x)\,d\xi = F(x) \qquad (0 \leqslant x \leqslant 1) \tag{5.1}$$

$$\int_x^\infty \xi^{-2\beta}\chi(\xi)\,J_\nu(\xi x)\,d\xi = G(x) \qquad (x > 1)$$

$$(K(\xi) \to 0,\ \xi \to 0)$$

which is applicable to many mixed poblems of the theory of elasticity, is reduced to the integral equation

$$\varphi(x) + \int_0^1 \varphi(y)\,dy \int_0^\infty K(t)\,J_\mu(tx)\,J_\mu(ty)\,dt = f(x) \tag{5.2}$$

$$(0 \leqslant x \leqslant 1,\ \mu = \nu + \beta - \alpha)$$

In order to reduce this integral equation to an infinite system we will construct its solution in the form of the series

$$\varphi(x) = x^\mu \sum_{m=0}^\infty \varphi_m P_m^{\mu,\,0}(1-2x^2) \tag{5.3}$$

in Jacobi polynomials. Making use of the orthogonality of the latter

$$\int_0^1 x^{2\mu+1} P_m^{\mu,\,0}(1-2x^2)\,P_n^{\mu,\,0}(1-2x^2)\,dx = \frac{\delta_{mn}}{2(1+\mu+2n)}$$

and also of the following relation [10]:

$$\int_0^1 \frac{x^{1+\mu}}{(1-x^2)^\nu} P_m^{\mu,-\nu}(1-2x^2)\,J_\mu(xy)\,dx = \frac{\Gamma(1-\nu+m)}{2^\nu m!\,y^{1-\nu}} J_\sigma(y)$$

$$(\operatorname{Re}\mu < -1,\ \operatorname{Re}\nu < 1,\ \sigma = 1-\nu+\alpha+2m)$$

we reduce the integral equation (5.2) to the infinite system

$$\frac{\varphi_n}{2(1+\mu+2n)}+\sum_{m=0}^{\infty}a_{nm}\varphi_m=f_n$$

$$a_{nm}=\int_0^{\infty}t^{-1}K(t)\,J_{1+\mu+2n}(t)\,J_{1+\mu+2m}(t)\,dt \qquad (5.4)$$

$$f_n=\int_0^1 x^{\mu+1}f(x)\,P_n^{\mu,0}(1-2x^2)\,dx$$

In the case $\mu=0$ the solution is obtained in the form of a series in Legendre polynomials.

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